

Borel hierarchies in infinite products of Polish spaces

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Abstract. Let H be a product of countably infinite number of copies of an uncountable Polish space X . Let Σ_ξ ($\bar{\Sigma}_\xi$) be the class of Borel sets of additive class ξ for the product of copies of the discrete topology on X (the Polish topology on X), and let $\mathcal{B} = \bigcup_{\xi < \omega_1} \bar{\Sigma}_\xi$. We prove in the Lévy–Solovay model that

$$\bar{\Sigma}_\xi = \Sigma_\xi \cap \mathcal{B}$$

for $1 \leq \xi < \omega_1$.

Keywords. Borel sets of additive classes; Baire property; Levy–Solovay model; Gandy–Harrington topology.

1. Introduction

Suppose X is a Polish space and N the set of positive integers. We consider $H = X^N$ with two product topologies: (i) the product of copies of the Polish topology on X , so that H is again a Polish space and (ii) the product of copies of the discrete topology on X . Define now the Borel hierarchy in the larger topology on H . To do so, we need some notation. An element of H will be denoted by $h = (x_1, x_2, \dots, x_n, \dots)$ and for $m \in N$, $p_m(h)$ will denote the first m coordinates, that is, $p_m(h) = (x_1, x_2, \dots, x_m)$. For $n \in N$ and $A \subseteq X^n$, $\text{cyl}(A)$ will denote the cylinder set with base A , that is,

$$\text{cyl}(A) = \{h \in H: p_n(h) \in A\}.$$

The Borel hierarchy for the larger topology on H can now be defined as follows:

$$\Sigma_0 = \Pi_0 = \{\text{cyl}(A): A \subseteq X^n, \quad n \geq 1\}$$

and for $\xi > 0$,

$$\Sigma_\xi = \left(\bigcup_{\eta < \xi} \Pi_\eta \right)_\sigma, \quad \Pi_\xi = \neg \Sigma_\xi.$$

The Borel hierarchy on H with respect to the smaller topology is defined in the usual way:

$$\bar{\Sigma}_1 = \{V: V \text{ is open in } H \text{ in the smaller topology}\}, \quad \bar{\Pi}_1 = \neg \bar{\Sigma}_1$$

and, for $\xi > 1$,

$$\bar{\Sigma}_\xi = \left(\bigcup_{\eta < \xi} \bar{\Pi}_\eta \right)_\sigma ; \quad \bar{\Pi}_\xi = \neg \bar{\Sigma}_\xi .$$

Let

$$\mathcal{B} = \bigcup_{\xi < \omega_1} \bar{\Sigma}_\xi = \bigcup_{\xi < \omega_1} \bar{\Pi}_\xi .$$

The problem we will address in this article is whether

$$\bar{\Sigma}_\xi = \Sigma_\xi \cap \mathcal{B} \quad \text{for } 1 \leq \xi < \omega_1 . \quad (*)$$

To tackle the problem we will use the methods of effective descriptive set theory. We therefore have to formulate the lightface version of $(*)$. We refer the reader to [Mo] and [L1] for definitions of lightface concepts. We take X to be the recursively presentable Polish space ω^ω hereafter.

Define

$$\Sigma_0^* = \Pi_0^* = \{ \text{cyl}(A) : A \text{ is } \Delta_1^1 \text{ in } (\omega^\omega)^n, n \geq 1 \},$$

and, for $1 \leq \xi < \omega_1^{ck}$,

$$\Sigma_\xi^* = \cup_1^1 (\cup_{\eta < \xi} \Pi_\eta^*)$$

and

$$\Pi_\xi^* = \neg \Sigma_\xi^* ,$$

where $\cup_1^1 (\cup_{\eta < \xi} \Pi_\eta^*)$ is a Δ_1^1 union of members of $\cup_{\eta < \xi} \Pi_\eta^*$. The lightface analogue of $(*)$ is then

$$\Sigma_\xi^* = \Delta_1^1 \cap \Sigma_\xi , \quad \text{for } 1 \leq \xi < \omega_1^{ck} . \quad (**)$$

In order to state the main result of the article, we equip ω^ω with the Gandy–Harrington topology, that is, the topology whose base is the pointclass of Σ_1^1 sets. The key property of this topology is that it satisfies the Baire category theorem (see [L1]). Consider now the following statement of set theory:

(O) Every subset of ω^ω has the Baire property with respect to the Gandy–Harrington topology.

The main result of the article can now be stated.

Theorem 1.1. Assume **(O)**. Let $1 \leq \xi < \omega_1^{ck}$. If A and B are Σ_1^1 subsets of H such that A can be separated from B by a Σ_ξ set, then A can be separated from B by a Σ_ξ^* set.

An immediate consequence is

COROLLARY 1.2.

(O) implies $(**)$.

The above results will be established in $\text{ZF} + \text{DC}$. Maitra *et al* [Ma] proved (*) for $\xi = 1$ in $\text{ZF} + \text{DC}$ by a boldface argument. We will provide a lightface argument in the Appendix for (**) when $\xi = 1$. Again this will be done in $\text{ZF} + \text{DC}$. Barua [Ba] proved Theorem 1.1 and Corollary 1.2. His proof was by induction on ξ . However, he left out the proof of the base step ($\xi = 1$). We will fill in the gap in this article. The proof of Theorem 1.1 presented here parallels very closely that of Louveau [L1], whereas the proof in [Ba] relies on the more abstract developments of [L2]. In consequence, the proof given here is somewhat simpler.

The paper is organized as follows. Section 2 is devoted to definitions and notation. Section 3 contains the detailed proof of Theorem 1.1 when $\xi = 1$, while §4 sketches how the proof of Theorem 1.1 can be completed by an inductive argument. In the concluding section, we will prove (*) under appropriate hypotheses and also mention open problems.

2. Definitions, notation and preliminaries

For $n \geq 1$, the Gandy–Harrington topology on $(\omega^\omega)^n$ will be denoted by T^n and the Gandy–Harrington topology on H will be denoted by T^∞ . Following Louveau [L1], we define for each ξ such that $1 \leq \xi < \omega_1^{ck}$ a topology T_ξ on H having for its base the pointclass $\Sigma_1^1 \cap \bigcup_{\eta < \xi} \Pi_\eta$.

Let \mathcal{S} be a second countable topology on $(\omega^\omega)^n$ (respectively, H). Let A be a subset of $(\omega^\omega)^n$ (respectively, H). By the *cosurrogate* of A we mean the largest \mathcal{S} -open set B such that $A \cap B$ is T^n -meager (respectively, T^∞ -meager). The *surrogate* of A is defined to be the complement of the cosurrogate of A . When \mathcal{S} is the topology T^n , we denote the surrogate (respectively, cosurrogate) of A by $\text{sur}^n(A)$ (respectively, $\text{cosur}^n(A)$). If $A \subseteq H$ and \mathcal{S} is the topology T_ξ , the surrogate (respectively, cosurrogate) of A will be denoted by $\text{sur}_\xi(A)$ (respectively, $\text{cosur}_\xi(A)$).

Lemma 2.1. *Let $m \geq 1$. If $A \subseteq (\omega^\omega)^m$ is T^m -open, then $\text{sur}^m(A)$ is the T^m -closure of A . Consequently, $\text{sur}^m(A) - A$ is T^m -nowhere dense.*

Proof. If B is Σ_1^1 and $A \cap B$ is T^m -meager, then $A \cap B$ must be empty, because $A \cap B$ is T^m -open and the Baire category theorem holds for T^m . Consequently, $\text{cosur}^m(A)$ is the union of basic open sets of the T^m -topology which are disjoint with A . It follows that $\text{sur}^m(A)$ is the T^m -closure of A . \square

Lemma 2.2. *Assume (O). Let $m \geq 1$. If $A \subseteq (\omega^\omega)^m$, then $A \Delta \text{sur}^m(A)$ is T^m -meager.*

Proof. Observe that ω^ω and $(\omega^\omega)^m$ are recursively isomorphic, so (ω^ω, T^1) and $((\omega^\omega)^m, T^m)$ are homeomorphic. Hence it follows from (O) that there is a T^m -open set B such that $A \Delta B$ is T^m -meager. So, if D is a Σ_1^1 subset of $(\omega^\omega)^m$, then $A \cap D$ is T^m -meager iff $B \cap D$ is T^m -meager, so that $\text{sur}^m(A) = \text{sur}^m(B)$. Since B is T^m -open, it follows from Lemma 2.1 that $\text{sur}^m(B) - B$ is T^m -nowhere dense, hence $B \Delta \text{sur}^m(B)$ is T^m -meager. Consequently, $A \Delta \text{sur}^m(A)$ is T^m -meager. \square

Note that the converse of Lemma 2.2 is true. Indeed, if $A \Delta \text{sur}^1(A)$ is T^1 -meager for every $A \subseteq \omega^\omega$, then, as is easy to verify, A has the Baire property with respect to T^1 for every $A \subseteq \omega^\omega$, that is, (O) holds.

3. The case $\xi = 1$

In this section we will prove Theorem 1.1 when $\xi = 1$.

Following [L1], we fix a coding pair (W, C) for the Δ_1^1 subsets of H , that is,

- (i) W is a Π_1^1 subset of ω ;
- (ii) C is a Π_1^1 subset of $\omega \times H$;
- (iii) the relations ' $n \in W \ \& \ C(n, h)$ ' and ' $n \in W \ \& \ \neg C(n, h)$ ' are both Π_1^1 ;
- (iv) for every Δ_1^1 subset A of H , there is $n \in W$ such that $A = C_n \stackrel{\text{def.}}{=} \{h \in H: C(n, h)\}$.

Define W_0 as follows:

$$\begin{aligned} m \in W_0 &\leftrightarrow m \in W \ \& \ (\exists n \geq 1)(\forall h)(\forall h')(C(n, h) \ \& \ p_n(h) \\ &= p_n(h') \rightarrow C(n, h')). \end{aligned}$$

Then W_0 is Π_1^1 . Indeed, W_0 is just the set of codes of Δ_1^1 cylinder subsets of H .

Lemma 3.1. *If A is a Σ_1^1 subset of H , then $\text{cl}_1(A)$ is Π_1 and Σ_1^1 , hence T_2 -open, where $\text{cl}_1(A)$ is the T_1 -closure of A .*

Proof. Indeed, for any A , $\text{cl}_1(A)$ is Π_1 , because it is a countable intersection of Π_1 sets. Now suppose A is Σ_1^1 . Then

$$\begin{aligned} h \notin \text{cl}_1(A) &\leftrightarrow (\exists n \geq 1)(\exists B)(B \text{ is a } \Sigma_1^1 \text{ subset of } (\omega^\omega)^n \ \& \ h \in \text{cyl}(B) \\ &\ \& \ A \cap \text{cyl}(B) = \emptyset) \\ &\leftrightarrow (\exists n \geq 1)(\exists B)(B \text{ is a } \Delta_1^1 \text{ subset of } (\omega^\omega)^n \\ &\ \& \ h \in \text{cyl}(B) \ \& \ A \cap \text{cyl}(B) = \emptyset). \end{aligned}$$

To prove the previous implication \rightarrow , let B be a Σ_1^1 subset of $(\omega^\omega)^n$ such that $h \in \text{cyl}(B)$ and $A \cap \text{cyl}(B) = \emptyset$. But then $p_n(A) \cap B = \emptyset$. Since $p_n(A)$ is Σ_1^1 , it follows from Kleene's separation theorem that there is a Δ_1^1 subset B' of $(\omega^\omega)^n$ such that $B \subseteq B'$ and $B' \cap p_n(A) = \emptyset$. Hence $h \in \text{cyl}(B')$ and $A \cap \text{cyl}(B') = \emptyset$, which establishes \rightarrow . Consequently,

$$h \notin \text{cl}_1(A) \leftrightarrow (\exists m)(m \in W_0 \ \& \ C(m, h) \ \& \ C_m \cap A = \emptyset).$$

So $\neg \text{cl}_1(A)$ is Π_1^1 . □

Lemma 3.2. *Assume (O). If A is a Π_1 subset of H , then $A \Delta \text{sur}_1(A)$ is T^∞ -meager.*

Proof. Choose subsets B_n of $(\omega^\omega)^n$, $n \geq 1$, such that

$$A = H - \bigcup_{n \geq 1} \text{cyl}(B_n).$$

Then

$$\begin{aligned} \text{sur}_1(A) - A &= \text{sur}_1(A) \cap \bigcup_{n \geq 1} \text{cyl}(B_n) \\ &\subseteq \bigcup_{n \geq 1} ([\text{sur}_1(A) \cap \text{cyl}(\text{sur}^n(B_n))] \\ &\quad \cup [\text{cyl}(B_n) - \text{cyl}(\text{sur}^n(B_n))]). \end{aligned}$$

Now

$$\text{cyl}(B_n) - \text{cyl}(\text{sur}^n(B_n)) = \text{cyl}(B_n - \text{sur}^n(B_n)).$$

The set on the right of the above equality is T^∞ -meager by virtue of Lemma 2.13 in [L2]. We will now prove that $\text{sur}_1(A) \cap \text{cyl}(\text{sur}^n(B_n))$ is T^∞ -nowhere dense. Note that $\text{sur}_1(A) \cap \text{cyl}(\text{sur}^n(B_n))$ is T_1 -closed, hence T^∞ -closed. Now let A' be a Σ_1^1 set contained in $\text{sur}_1(A) \cap \text{cyl}(\text{sur}^n(B_n))$. Then

$$\text{cyl}(p_n(A')) \subseteq \text{cyl}(\text{sur}^n(B_n)).$$

Hence

$$\begin{aligned} A \cap \text{cyl}(p_n(A')) &\subseteq \text{cyl}(\text{sur}^n(B_n)) - \text{cyl}(B_n) \\ &= \text{cyl}(\text{sur}^n(B_n) - B_n). \end{aligned}$$

Consequently, by virtue of Lemma 2.2 and Lemma 2.13 in [L2], $A \cap \text{cyl}(p_n(A'))$ is T^∞ -meager. Since $\text{cyl}(p_n(A'))$ is T_1 -open, it follows that $\text{cyl}(p_n(A')) \subseteq \text{cosur}_1(A)$. Hence A' is empty because A' is also contained in $\text{sur}_1(A)$. Thus $\text{sur}_1(A) \cap \text{cyl}(\text{sur}^n(B_n))$ is T^∞ -nowhere dense. It follow from (1) that $\text{sur}_1(A) - A$ is T^∞ -meager. Since $A - \text{sur}_1(A)$ is easily seen to be T^∞ -meager, we are done. \square

Lemma 3.3. *If A and B are Σ_1^1 subsets of H such that A can be separated from B by a Σ_1 set, then $A \cap \text{cl}_1(B) = \emptyset$.*

Proof. Suppose D is a Π_1 subset of H such that $A \cap D = \emptyset$ and $B \subseteq D$. Hence, by Lemma 3.2, $B - \text{sur}_1(D)$ is T^∞ -meager. But $B - \text{sur}_1(D)$ is T^∞ -open, so $B \subseteq \text{sur}_1(D)$.

Since $\text{sur}_1(D)$ is T_1 -closed, $\text{cl}_1(B) \subseteq \text{sur}_1(D)$. Now $A \cap \text{sur}_1(D)$ is T^∞ -meager, so $A \cap \text{cl}_1(B)$ is T^∞ -meager. By Lemma 3.1, $A \cap \text{cl}_1(B)$ is Σ_1^1 , hence $A \cap \text{cl}_1(B)$ must be empty. \square

Lemma 3.4. *If A and B are Σ_1^1 subsets of H such that $A \cap \text{cl}_1(B) = \emptyset$, then A can be separated from B by a Σ_1^* set.*

Proof. Define

$$P(h, n) \leftrightarrow h \notin A \vee (n \in W_0 \ \& \ C(n, h) \ \& \ C_n \cap B = \emptyset).$$

Then P is Π_1^1 and $(\forall h)(\exists n)P(h, n)$. By Kreisel's selection theorem [Mo], there is a Δ_1^1 -recursive function $f: H \rightarrow \omega$ such that $(\forall h)P(h, f(h))$. Let

$$D = \{n \in \omega: n \in W_0 \ \& \ C_n \cap B = \emptyset\}.$$

Then D is Π_1^1 and $f(A) \subseteq D$. Since $f(A)$ is Σ_1^1 , there is a Δ_1^1 set $E \subseteq \omega$ such that $f(A) \subseteq E \subseteq D$. Let

$$R(h, n) \leftrightarrow n \in E \ \& \ C(n, h),$$

Then R is Δ_1^1 , because if

$$R'(h, n) \leftrightarrow n \in E \ \& \ \neg C(n, h),$$

then both R and R' are Π_1^1 , $R \cap R' = \emptyset$ and $R \cup R' = H \times E$. Set

$$G_n = \{h: R(h, n)\}, \ n \in \omega.$$

Then $\cup_{n \geq 0} G_n$ is a Σ_1^* set which separates A from B . \square

Lemmas 3.2, 3.3 and 3.4 establish Theorem 1.1 for $\xi = 1$.

4. Proof of Theorem 1.1

The proof of Theorem 1.1 is by induction on ξ . So we fix $\xi > 1$ and assume Theorem 1.1 is true for all $\eta < \xi$. Lemmas 3.1–3.4 can be formulated and proved at level ξ , thereby completing the proof of Theorem 1.1 at level ξ . We omit the proofs because they are exactly like the proofs of Lemmas 7, 8, 9 and Theorem B in [L1].

We observe that the inductive hypothesis that Theorem 1.1 hold at all levels $\eta < \xi$ is by itself not sufficiently strong to prove the analogue of Lemma 3.2 at level ξ and hence the theorem itself at that level. For this we need that analogues of Lemma 3.2 hold at all levels $\eta < \xi$. It is at this point in the proof that assumption (O) is needed to ensure that Lemma 3.2 hold at level $\xi = 1$, the higher levels of Lemma 3.2 then being proved by inducting up from the base level.

5. Concluding remarks

For $\alpha \in \omega^\omega$, we now consider the following statement of set theory:

(α) Every subset of ω^ω has the Baire property with respect to the topology whose base is the pointclass of $\Sigma_1^1(\alpha)$ sets.

It is straightforward to relativize Theorem 1.1 to α under the assumption that (α) holds. The next result is provable in $\text{ZF} + \text{DC} + (\forall \alpha)((\alpha))$.

Theorem 5.1. *Let X be an uncountable Polish space and let $H = X^N$. Then, for $1 \leq \xi < \omega_1$,*

$$\bar{\Sigma}_\xi = \Sigma_\xi \cap \mathcal{B}.$$

Under the assumption that there is an inaccessible cardinal, Solovay [S] proved that $\text{ZF} + \text{DC}$ holds in the Lévy–Solovay model. Furthermore, it was observed by Louveau (p.43 of [L2]) that the statement $(\forall \alpha)((\alpha))$ holds as well in the model.

Whether Theorem 5.1 is provable in ZFC remains an open problem. Indeed, we do not have an answer to the problem even when $\xi = 2$.

It is not difficult to prove that the axiom of determinacy implies $(\forall \alpha)((\alpha))$ so that Theorem 5.1 is provable in $\text{ZF} + \text{AD}$ (see [Mo]). On the other hand, the axiom of choice implies $\neg(\text{O})$ in ZF.

Appendix

We will now prove Theorem 1.1 for $\xi = 1$ without assuming (O). In view of Lemma 3.4, it will suffice to prove that $A \cap \text{cl}_1(B) = \emptyset$. Define

$$P(h, n) \leftrightarrow (n \geq 1) \ \& \ (\exists h')(p_n(h)h' \in B),$$

where $p_n(h)h'$ is the catenation of $p_n(h)$ and h' . Note that P is Σ_1^1 . Let

$$h \in \bar{B} \leftrightarrow (\forall n \geq 1)P(h, n),$$

so that \bar{B} is the closure of B in the product of discrete topologies on H . Consequently, $\bar{B} \subseteq H - A$. Define

$$Q(h, n) \leftrightarrow (n \geq 1) \ \& \ (\neg P(h, n) \vee h \notin A).$$

Then Q is clearly Π_1^1 and $(\exists n)Q(h, n)$. So there is a Δ_1^1 -recursive function $f: H \rightarrow \omega$ such that $(\forall h)Q(h, f(h))$. Let

$$S(h, n) \leftrightarrow (n \geq 1) \ \& \ (f(h) \neq n \vee h \notin A).$$

Claim.

- (i) S is Π_1^1 ,
- (ii) $(\forall h)(\forall n \geq 1)(P(h, n) \rightarrow S(h, n))$,
- (iii) $h \notin A \leftrightarrow (\forall n \geq 1)S(h, n)$.

To see (ii), assume $P(h, n)$. Then we must have $h \in A \rightarrow f(h) \neq n$. Hence $S(h, n)$. For (iii), suppose $h \notin A$. Clearly, then $(\forall n \geq 1)S(h, n)$. Suppose now that $h \in A$. Then there is n such that $f(h) = n$, hence $\neg S(h, n)$. (iii) now follows.

Now turn each S_n into a cylinder set as follows. Define

$$R(h, n) \leftrightarrow (\forall h')S(p_n(h)h', n),$$

so R is Π_1^1 . Note that P_n and R_n are cylinder sets, that is,

$$P(h, n) \ \& \ p_n(h) = p_n(h') \rightarrow P(h', n)$$

and

$$R(h, n) \ \& \ p_n(h) = p_n(h') \rightarrow R(h', n).$$

Claim. $(\forall h)(\forall n)(P(h, n) \rightarrow R(h, n))$.

So suppose $P(h, n)$. Then, for every h' , $P(p_n(h)h', n)$, hence $S(p_n(h)h', n)$, so $R(h, n)$.

To complete the proof, let $h \in A$. Then there is $n \geq 1$ such that $\neg S(h, n)$, hence $\neg R(h, n)$. Now $\neg R_n$ is Σ_1^1 and Π_0 because R_n is a cylinder set. Moreover, $\neg R_n \cap B = \emptyset$ because $\neg R_n \subseteq \neg P_n$ and $\neg P_n \cap B = \emptyset$. Hence $\neg R_n$ is a T_1 -open set containing h and disjoint from B . So $h \notin \text{cl}_1(B)$.

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